A PROPERTY OF PLANAR CONVEX BODIES

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ABSTRACT

It is proved that from each interior point of a planar convex body emanate three distinct vectors terminating on the boundary, the sum of any two of which also terminates in the boundary. Some other related results are obtained.

In this article we prove that from each interior point of a planar convex body emanate three distinct vectors terminating on the boundary, the sum of any two of which also terminates on the boundary. (See Figure 1). We also show the related result that through each interior point of a planar convex body pass the boundaries of three distinct translates which cover the body in the sense of property β below. (See Figure 2). In addition, we investigate the nature of the set of points on the boundary which can be end points of such triples of vectors and their pair sums. Finally, we pose some related unsolved problems.

We begin by saying that a planar body C has:*

property α at $p \in C^\circ$ if there exist distinct points $x, y, z \in BdC$ such that $(x-p)+(y-p), (x-p)+(z-p)$ and $(z-p)+(y-p)$ also belong to BdC. (See Figure 1).

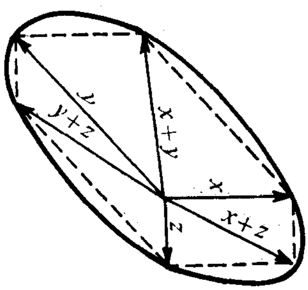

Figure 1

property β at $p \in C^\circ$ if there exist distinct translates, C_1, C_2, and C_3, such tha $\bigcap_{i=1}^{3} BdC_i = \{p\}$ and the set $BdC \cap BdC_i \cap BdC_j$ for each $i \neq j$ consists of exactly one point. (See Figure 2).

There exists a natural relationship between the three points of property α and the three translates of property β as given by the following lemma:

LEMMA 1. (1) *If property α is satisfied at $p \in C^\circ$ with x, y, and z, then property β is satisfied with the translates $C+(p-x)$, $C+(p-y)$ and $C+(p-z)$.*

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* The symbols C° and BdC denote the interior and boundary of C respectively.

(2) *If C is rotund (i e, there are no line segments in BdC) and property β is satisfied with the translates $C + a_i (i = 1,2,3)$, then property α is satisfied with the points $p - a_i (i = 1, 2, 3)$.*

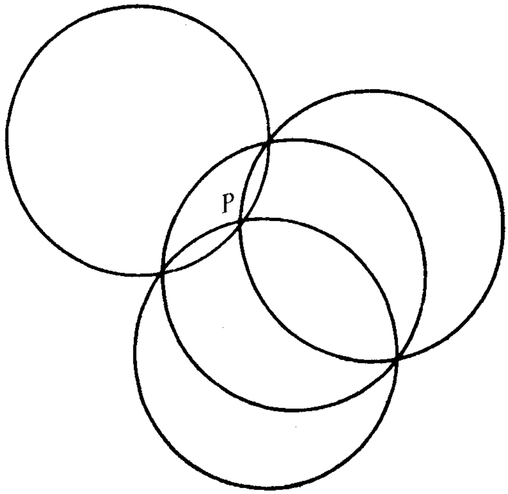

Figure 2

Proof. The proof of (1) is obvious. For the proof of (2), let us assume that $p = 0$ and that $C + a_1$, $C + a_2$ and $C + a_3$ are in consecutive counterclockwise order around BdC. Define x_i by $\{x_i\} = BdC \cap Bd(C + a_i) \cap Bd(C + a_{i+1})$. Then to obtain property α it clearly suffices to show that $x_i = -a_{i+2}$ (mod. 3). Without loss of generality we can assume that we have $x_2 \leqq -a_1 < x_1 \leqq -a_3$ or $-a_1 \leqq x_2 < -a_3 \leqq x_1$, where the order is taken to be counterclockwise around BdC. Consider the first case (the second case will be similar) and the resulting two pairs of two parallel and equal (in length) line segments $[0, -a_3]$ and $[x_2, x_2 - a_3]$; $[0, -a_1]$ and $[x_1, x_1 - a_1]$. From the convexity of C and the absence of line segments in BdC it follows that these four segments must form a parallelogram, in which case $x_1 = -a_3$ and $x_2 = -a_1$. This in turn forces $x_3 = -a_2$, proving the Lemma.

It is easily seen that the relationship of part (2) in the Lemma need not hold for non-rotund bodies (e.g., a parallelogram). Now we prove our main result:

THEOREM 1. *Properties α and β are satisfied at each interior point of a planar convex body.*

Proof. By Lemma 1 it suffices to show only property α at each interior point. We will first prove this for a rotund body and then pass to the limit for the general case.

Suppose C is rotund and that $p = 0 \in C^{\circ}$. Let x_1 be an arbitrary point in BdC. Then there will be exactly two chords of C equal and parallel to the segment

$[0,x_1]$. Taking the chord on the counterclockwise side of x_1 we will obtain a $y_1 \in BdC$ such that $x_1 + y_1 \in BdC$. Repeating this process (counterclockwise around BdC) we also obtain a $z_1 \in BdC$ such that $y_1 + z_1 \in BdC$, and a $x_2 \in BdC$ such that $z_1 + x_2 \in BdC$. If $x_1 = x_2$, then we have our desired triple of points. Otherwise we continue this process to obtain sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ such that for each n, $x_n + y_n$, $y_n + z_n$ and $z_n + x_{n+1}$ belong to BdC.

Now, since $x_1 \neq x_2$ and C is not a parallelogram, we obtain two cases: either $x_1 < x_2 < z_1$ or $y_1 < x_2 < x_1$.

Case I. In case $x_1 < x_2 < z_1$, let us prove by induction that for each k

$$\begin{aligned} & x_1 \leqq x_k \leqq x_{k+1} < z_1 \\ & y_1 \leqq y_k \leqq y_{k+1} < x_1 \\ \text{and}\quad & z_1 \leqq z_k \leqq z_{k+1} < y_1 . \end{aligned}$$

For $k = 1$ we have $x_1 < x_2 < z_1$ by assumption and the inequalities $y_1 < y_2 < x_1$ and $z_1 < z_2 < y_1$ follow from this. Now assume that the above proposition holds for all $i \leqq k$. Then from the fact that $z_1 \leqq z_{k+1} < y_1$ it clearly follows that $x_1 \leqq x_{k+1} \leqq x_{k+2} < z_1$. From this in turn follow the other two inequalities $y_1 \leqq y_{k+1} \leqq y_{k+2} < x_1$ and $z_1 \leqq z_{k+1} \leqq z_{k+2} < y_1$, which proves the proposition for $k + 1$. Hence, each of the three sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ is a monotonic sequence contained in a closed proper subarc of BdC. Therefore, the sequences will converge to distinct points x, y and z respectively. Taking limits we also obtain $x + y$, $x + z$ and $y+z \in BdC$. Thus, property α is satisfied in Case I

Case II. In case $y_1 < x_2 < x_1$ we first need to prove the following lemma:

LEMMA 2. *If there are three consecutive translates of the rotund body C passing through $p \in C^{\circ}$ (i.e., there exist translates C_1, C_2 and C_3 such that $\bigcap_{i=1}^{3} BdC_i = \{p\}$ and $BdC_i \cap BdC_{i+1} \cap BdC \neq \Delta$ for $i = 1,2$) which fail to cover BdC (i.e., $BdC - \bigcup_{i=1}^{3} C_i \neq 0$), then property β is satisfied at p.*

Proof. Suppose $p = 0$ and C_1 is the first translate in the counterclockwise direction from the arc $BdC - \bigcup_{i=1}^{3} C_i$. For $x \in BdC$ let $C(x)$ be the first translate of C in the counterclockwise direction from x which has $[0, x]$ as a chord. Clearly each translate can be so expressed. Also let x' denote the other point of intersection of BdC with $BdC(x)$. Let a_1 be such that $C_1 = C(a_1)$. Put $b_1 = a_1'$ and $c_1 = b_1'$. Then $C_2 = C(b_1)$ and $C_3 = C(c_1)$. Then put $a_2 = c_1'$, $b_2 = a_2'$ and $c_2 = b_2'$. Continuing in this way we obtain sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ such that $a_{n+1} = c_n'$, $b_{n+1} = a_{n+1}'$ and $c_{n+1} = b_{n+1}'$. The assumption that C_1, C_2 and C_3 fail to cover BdC means that $a_1 < a_2 < c_1$. Now (using the fact that each two distinct translates of C whose boundaries contain 0 intersect in two distinct arcs in BdC, no one of which is contained in the other) employing the same argument as we did in Case I we see that the sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ converge to distinct a, b, c respectively. Consequently, the three sequences of translates $\{C(a_n)\}_{n=1}^{\infty}$, $\{C(b_n)\}_{n=1}^{\infty}$ and $\{C(c_n)\}_{n=1}^{\infty}$ converge to the translates

$C(a)$, $C(b)$ and $C(c)$ respectively which obviously satisfy property β, finishing the proof of Lemma 2.

Now consider the three translates $C-x_2$, $C-y_1$, and $C-z_1$. Then $\{z_1\}=Bd(C-x_2)\cap Bd(C-y_1)$ and the two points of $BdC\cap Bd(C-z_1)$ lie outside the arc $BdC\cap((C-x_2)\cup(C-y_1))$. Then clearly $C-x_2$, $C-y_1$ and $C(x_1)$ will constitute three consecutive translates passing through 0 which fail to cover BdC. Hence, by Lemma 2, property β, and therefore property α by Lemma 1, is satisfied at 0. Thus, property α holds in Case II.

Now we consider a general convex body C and let $p=0\in C^{\circ}$. Then we can easily find a sequence of rotund bodies $\{C_n\}_{n=1}^{\infty}$ such that $0\in C_n^0\subseteq C$ for each n and C is the limit of $\{C_n\}_{n=1}^{\infty}$ in the Hausdorff metric. For each n there exist x_n, y_n *and* z_n in BdC for which property α holds with respect to C_n. Since C is sequentially compact, we can then find a subsequence k such that $\{x_{k_n}\}_{n=1}^{\infty}$, $\{y_{k_n}\}_{n=1}^{\infty}$ and $\{z_{k_n}\}_{n=1}^{\infty}$ converge to points x, y, z respectively in BdC. Moreover, x, y and z are all distinct. For suppose $x=y$, then since $0\in$ arc $(x_{k_n}, y_{k_n})\subseteq BdC_{k_n}$ we have, by taking limits, $0\in BdC$, a contradiction. Also by taking limits we have $x+y$, $y+z$ and $x+z$ in BdC. Therefore, property α is satisfied with x, y and z, which finishes the proof of the Theorem.

If property α is satisfied at 0 by a triple x, y and z, then x, $x+y$, y, $y+z$, z and $x+z$ determine an inscribed centrally symmetric hexagon. (See Figure 1)*. Let us denote by V the set of all points on BdC which are vertices of some such inscribed centrally symmetric hexagon. The set V may not be all of BdC; for example, the vertices of a triangle are not in V. In the case C is rotund and differentiable, it seems reasonable to conjecture that $V=BdC$. However, the best we have done in trying to settle this is the following theorem:

THEOREM 2. *Suppose C is rotund and differentiable. Then* (1) *$BdC-V$ consists of isolated points* (*and hence is countable*) *and*

(2) *if $x\in BdC-V$, then $x'\in V$, where x' is the unique point z so that $[x,z]$ is a diameter of C.*

Proof. Let x be any point in BdC amnd let U_x be an open arc of BdC which contains x, but small enough so that no diameter of C intersects it twice. Let y be to the clockwise side of x in U_x. Denote by $[a,b]$ the other chord of C equal and parallel to $[x,y]$. Then we have nine possible cases:

1. $a<y'<b<x'$	5. $y'<a<b=x'$
2. $y'<a<x'<b$	6. $a<y'<b=x'$
3. $a<y'<x'<b$	7. $y'=a<b=x'$
4. $y'<a<b<x'$	8. $y'=a<b<x'$
	9. $y'=a<x'<b$.

* It is well known that in each convex body one can inscribe an affine-regular (hence, centrally symmetric) hexagon. For references to this and other results on such inscribed hexagons see Grünbaum [1; p. 242].

Correspondingly, we get nine cases if y is to the counterclockwise side of x in U_x. Now we need three lemmas:

LEMMA 3. *Suppose one of the first four cases prevails. Then* $x \in V^\circ$ *and* $y \in V^\circ$.

Proof. Suppose Case I holds and that y is to the clockwise side of x in U_x. The other cases will be similar. Consider the two arcs arc (x,y) and arc (b,a) in BdC. Next "reflect" one upon the other so that a coincides with y. Considering the direction of the support lines at x, y, x', y', a and b, we see that arc (x,y) must intersect the reflection of arc (b,a) in a point other than x and y. Then we obtain in the obvious way an inscribed hexagon with opposite sides equal and parallel. By the rotundity of C, its vertices, including x and y, must belong to V. Now if we consider x fixed, there will obviously be a neighborhood W of y such that the Case I configuration holds relative to x and any point in W. Hence, $y \in V^\circ$. Now fixing the y, we can find a neighborhood W of x such that the Case I configuration holds with respect to any point in W and y. Hence, $x \in V^\circ$, which finishes the proof of the Lemma.

LEMMA 4. *Suppose there are three distinct* y *in* U_x *for which either one of cases, 5, 6 or 7 prevails (i.e.,* $b = x'$*). Then* $x \in V$.

Proof. We can assume without loss of generality that two of these points, say y_1 and y_2, are to the clockwise side of x. Then it is easily seen that x, y_1, y_2 and x' comprise four vertices of an inscribed centrally symmetric hexagon.

LEMMA 5. *Suppose there exists an open arc* N *containing* x *such that for any* $y \in N - \{x\}$, *cases 8 or 9 prevail. Then there exists an open arc* M *containing* x *such that* $M - \{x\} \subseteq V$.

Proof. Let $A = \{y \in N - \{x\} : y' < b < x'\}$ and $B = \{y \in N - \{x\} : y' < x' < b\}$. Let $y \in A$. Then reflecting arc(y,x') upon the appropriate opposite arc we get $y \in V$. Moreover, the configuration of case 8 will prevail in a small neighborhood of y. Hence, A is open and contained in V°. Now if $y \in B$, the configuration of case 9 prevails in a small neighborhood of y, so that B is open, too. Therefore, if arc(x,z) $\subseteq N$, then arc (x,z) is contained entirely in A or B. If arc$(x,z) \subseteq B$, then for $x < y < z$ let $[e(y), f(y)]$ be the other chord which is equal and parallel to $[y', x']$. Clearly, e is a continuous function of y. And if we reflect arc(y', x') upon arc$(e(y), f(y))$ we get $e(y) \in V$ with $x < e(y) < y$. Hence, the range of e, restricted to arc(x,z), must be an interval of the form arc(x,w) which is contained wholly in V. Similarly, we take care of the case when arc $(z,x) \subseteq N$. Hence, there exists an open arc M such that $M - \{x\} \subseteq V$.

Now for the proof of part (1) of the Theorem we let x be any fixed point not in V. Then by applying Lemmas 3 and 4 we can find a neighborhood W of x such that for any $y \in W$ case 8 or 9 prevails. Then applying Lemma 5 we can find a deleted neighborhood of x which is contained in V. Hence, the points of $BdC - V$ are isolated and consequently, $BdC - V$ is countable.

For part (2), let $x \in BdC - V$ and choose y to the clockwise side of x in U_x. Then case 8 or 9 prevails. In case 8, reflecting the arc(y, x') upon the appropriate arc(c, d) we get $x' \in V$. And in case 9 we reflect arc(a, x') upon the appropriate arc(c, d) to get $x' \in V$. Hence, $x \in BdC - V$ implies $x' \in V$, finishing the proof of the Theorem.

There remain some interesting and seemingly difficult unsolved problems relating to the nature of V in BdC. Namely, (1) is $V = BdC$ when C is rotund and differentiable; and (2) in the general case, is V dense in BdC or is $BdC - V$ even countable?

Also property α can be generalized to any closed curve C in the plane to: C has property α at p if there exist distinct x, y, z in C such that $(x-p)+(y-p)$, $(x-p)+(z-p)$ and $(y-p)+(z-p)$ are also in C. Some unsolved problems are (3) is there such a point for each closed curve C; (4) if so, what is the nature of such points; and (5) can convex closed curves be characterized as those closed curves having property α at each interior point?

REFERENCES

1. Grünbaum, B., 1963, Measures of symmetry for convex sets, in: *Convexity*, Vol. 7 of *Proc. of Symposia in Pure Math.* A.M.S., pp. 233–270.

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